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SURVEY OF THE NUMERICAL SOLUTION TECHNIQUES FOR VARIATIONAL PROBLEMS IN THE DYNAMICS OF FLIGHT *13

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Indirect methods of numerical solution of variational problems in rocket flight are reviewed, covering all Russian literature data. Local methods of continuous search are discussed, including a modified version of Newton's method, variants of the steepest descent method, solutions of variational problems with a free right end of the trajectory. Comparisons of the various methods are given, including the example of asymmetric flight to Mars with return, resulting in a 20% advantage over symmetrical flight. A minimum value N_{min} is derived at which the flight will be accomplished at maximum thrust. author

The methods of numerical solution of variational problems are divided into indirect and direct. In the indirect methods, the boundary-value problems are solved for differential equations describing the optimal motion of the object involved. The purpose of the indirect methods is the gradual improvement of a certain set of parameters (deficient boundary conditions of the boundary-value problem). The direct methods consist in optimizing the functionals by successive improvement of the control functions.

The purpose of the present paper is mainly to survey certain indirect methods. The review covers primarily material published in the USSR.

Of the local methods of continuous search we consider Newton's method with modifications proposed by us (including considerations on the correlation of the modified Newton method with the gradient methods), a version of the method of steepest descent, methods of solving variational problems with a free right end of the trajectory, and methods for linear systems.

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INTRODUCTION

The optimum programming problem arises each time an attempt is made to construct a scientifically justified system or to determine the maximum possibilities of existing systems. The problem is to single out, from all possible programs for the control variables, the specific program which will yield the maximum (or minimum) of some quantity at the end of the control process and which, at the same time, will satisfy the prescribed boundary conditions.

The methods of numerical solution of variational problems are divided into indirect and direct. The indirect methods consist in the solution of the boundary problems for differential equations describing the optimum motion of the object involved. The direct methods consist of methods of maximizing (or minimizing) functionals. While the purpose of the indirect methods consists in the gradual improvement of some set of parameters (deficient initial conditions of the boundary-value problem), the purpose of the direct methods is the successive improvement of the control functions, which means that direct methods belong to the class of functional methods. /5

Let the specified dynamic system be described by the equations

$$\dot{x}_i = f_i(x_1, \ldots, x_n; u_1, \ldots, u_r; t) \tag{1}$$

or, in vector form:

$$\dot{x} = f(x, u, t), \quad 0 \leq t \leq T. \tag{1a}$$

Required: to find the equation $u = (u_1, \ldots, u_r)$ which, in the time T, will transform the system (1) from its prescribed initial position

$$x(0) = \{x_1^0, \ldots, x_n^0\} \tag{2}$$

to a prescribed final position such that a certain functional S assumes its maximum (or minimum) value under restrictions imposed on the control

$$u \in U. \tag{3}$$

Condition (3) makes the problem nonclassical. The general apparatus for solution of such a problem is the principle of the maximum (Bibl.1, 2). Special modifications of the classical methods (Bibl.3 - 5) which are useful in solving /6 certain problems of the form of eq.(1) or eq.(3) should also be mentioned. As a branch of the nonclassical calculus of variations, the method of dynamic programming (Bibl.6 - 7) occupies a special position in this respect.

The object of the present work is a general survey, primarily of certain indirect methods. The review will cover mainly materials published in the USSR.

I. METHODS OF SOLVING VARIATIONAL PROBLEMS OF FLIGHT DYNAMICS REDUCED TO BOUNDARY PROBLEMS

1. General Outline of Indirect Methods

By means of the maximum principle, the variational problem [eqs.(1)-(3)] is reduced to a system of the order 2n

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i} \quad (i = 1, \ldots, n), \tag{4}$$

where

$$H = \sum_{i=1}^{n} p_i f_i (x, u, t), \tag{5}$$

In eqs.(4) the control u is replaced by the function $u = u(x, p, t)$ obtained from the condition that the function $H = H(u)$ is to be minimum (or maximum), with consideration of the condition (3).

If eqs.(4) are integrated, the boundary-value problem reduces to a system of algebraic or transcendental equations. If this is not the case, the solution of the boundary-value problem reduces to a successive solution of Cauchy problems, and the problem will consist in organizing the selection of the corresponding lacking initial conditions. /7

In the arbitrary case, it is required to solve the problem of finding the zeros of the system of equations*:

$$\varphi_i \equiv \varphi_i (y_1, \ldots, y_m) = 0 \quad (i = 1, \ldots, m). \tag{6}$$

The restriction (3) is automatically taken into account in calculating eq.(6) by means of eqs.(4). We note that, with this approach, the problem with

* In the problem (1) - (3), in the general case, $y = (y_1, \ldots, y_n) = (p_1^0, \ldots, p_n^0)$.

fixed T and the problem with unfixed T are both solved in the same manner (Bibl.8).

We will characterize the iterative process by the value of the function

$$\Phi = \sum_{i,k=1}^{n} a_{ik} \varphi_i(y) \varphi_k(y), \tag{7}$$

where a_{ik} are the constant coefficients of a positive quadratic form of determinate sign. Instead of eq.(7) we may use:

$$\psi = \psi\left(\sum_{i,k=1}^{n} a_{ik} \varphi_i(y) \varphi_k(y)\right), \tag{8}$$

where ψ is a continuous, monotonous, continuously differentiable function ϕ such that $\psi \geq 0$, and such that it vanishes only if ϕ vanishes.

In accordance with another paper (Bibl.9), the methods of finding the extremum (here, the Zero) of the function of the residue ϕ (which is sometimes called the output or estimate function) may conveniently be classified into three groups.

The first group includes the so-called methods of blind search (random sampling, homeostat principle, scanning), which do not use information accumulated during the preceding stage. /8

The second group includes the methods of continuous search [termed "local search" by Gel'fand (Bibl.9)]. These methods consist essentially in the continuous motion of the working point in parameter space in some selected manner directly to the point at which ϕ reaches a minimum value. From the obtained point a new direction is determined, along which the descent is again continued to the local minimum of ϕ, and so on. If the search is conducted in directions which, once and for all, are fixed in a definite order, then we have an analog of the Gauss-Seidel method. In the method of steepest descent (Bibl.10, Chapter III) and the gradient method, the direction of search is determined from the direction of the gradient ϕ. The schemes for the latter two methods closely resemble each other. It has been noted (Bibl.11, 12) that these two methods permit consideration of any restrictions on the phase coordinates; for great deviations from the minimum it is preferable to use the method of steepest descent, while the gradient method is better for small deviations. One feature of the variational problems of rocket dynamics is that, when the system (4) is put into dimensionless form, the process of solution of the problem may in some cases be considered completed when the working point falls within the region $\phi < 10^{-8} - 10^{-9}$. The most effective method in this case is that of Newton (Bibl.10, Chapter IV) which converges more rapidly than the gradient method in the neighborhood of small ϕ. The methods of this group may be made still more /9
effective by the introduction of a memory, so as to determine the direction of

search with allowance for the points of the provisional minimum of the function ϕ found in the preceding stages of the iteration. Another paper (Bibl.13) describes one algorithm of this kind for a nonlocal method of continuous search. According to this algorithm, the working point moves successively in substantially different (independent) directions.

There is a special group, consisting of nonlocal search methods (Bibl.9) in which the process of displacement of the working point in parameter space ceases to be continuous. On "cyclization" of the working point, motion takes place along the bottom of a "ravine", i.e., along the straight line connecting two points of the relative extremum, found by the method of coarse descent from two different points in parameter space. The length of the ravine step is taken considerably greater than that of the gradient step. From the point so found, a coarse descent is again performed, the direction of the ravine is refined, and a ravine step is again taken. In some cases, the "ravine method" is effective and permits investigation of a broad region of small values of the function of the residue ϕ.

2. The Newton Method

We have already noted that the Newton method is one of the most effective methods for the numerical solution of boundary-value problems. The modification of that method proposed by us has proved highly useful (Bibl.8, 14). One of the possible causes of divergence of the method is that at some stages of /10 the iteration the value of a single step, in the region of large values of ϕ, is too great. This is because the Newton method, like any local method, utilizes only the information on the initial point of the iteration. On the basis of this information, which gives essentially an idea of the function in a narrow neighborhood of the initial point, the position of the zero is predicted and the next step is then taken in the direction so found.

The accuracy of the Taylor expansion, however, decreases with increasing distance from the initial point.

In some cases, the following method may help to overcome contradictions of this kind. We will introduce a correction at certain stages of the search. Of course, this will not make it possible to get along altogether without additional information on the course of the process, although the amount of such information should be kept to a minimum insofar as possible.

We will introduce the correction depending on the distribution of the value of the residue in the boundary conditions at discrete points of the ray connecting two successive points of the iteration. Consider the "gross error" (residue) function:

$$\Phi(k,\alpha) = \sqrt{\sum_{i=1}^{n} \varphi_i^2(y_{k-1} + \alpha\eta_{k-1})}\,;$$
$$\Psi[\Phi(k,\alpha)] = \Psi[\Phi(y_{k-1} + \alpha\eta_{k-1})]. \qquad (9)$$

where k is the number of the iteration step; α is a parameter, $0 \leq \alpha \leq 1$ increasing linearly along the length of the ray $[y_{k-1}, y_k]$; τ_k is the length of the k-th step of the iteration. At $\alpha = 0$ and $\alpha = 1$, eq.(9) gives the residue for the (k-1)th and the k-th step, respectively: $\phi(k, 0) = \phi(k-1, 1)$ and $\phi(k,1)$. On the appearance, at the k-th step, of signs of divergence of the Newton /11 scheme: $\phi(k, 0) \leq \phi(k, 1)$, we supplement the classical scheme by calculating the value of the residue at one or several points of the ray $[y_{k-1}, y_k]$. The choice of the number of N_k and the disposition of α_j ($j = 1, \ldots, N_k$) optimum for convergence, is a complex and unsolved problem.

The following method has proved in practice to be rather universal. Selecting $\alpha = \frac{1}{2}$, we calculate the residue $\phi(k, \frac{1}{2})$ in the center of the ray. If $\phi(k, 0)$, $\phi(k, \frac{1}{2})$ and $\phi(k, 1)$ form a downwardly convex function of a discrete argument, then we can lay, through three points, the parabola approximating $\phi(k, \alpha)$ and find the point of the minimum α^* of the Lagrange interpolation polynomial:

$$\alpha^* = \frac{3\phi(k,0) - 4\phi(k,\frac{1}{2}) + \phi(k,1)}{4[\phi(k,0) - 2\phi(k,\frac{1}{2}) + \phi(k,1)]} \qquad (10)$$

If the condition of downward convexity is not satisfied or if $\alpha^* \leq 0$, the above-described operations are repeated, but this time on only half of the initial ray, $0 < \alpha \leq \frac{1}{2}$, etc.

Approximation of a parabola is obviously not the only method of performing a correction depending on the results of a "probing" of the "interiority of the step". The following elementary algorithm has also proved useful in practice:

1) A "classical step" is taken; 2) $\phi(k, 0)$ and $\phi(k, 1)$ are calculated; /12 3) if the condition $\phi(k, 1) < \phi(k, 0)$ is satisfied, the classical scheme is retained; if it is not satisfied, $\phi(k, \frac{1}{2})$ is calculated; 4) if the condition $\phi(k, \frac{1}{2}) < \phi(k, 0)$ is satisfied, then the Newton scheme is again utilized; if it is not satisfied, then $\phi(k, \frac{1}{4})$ is found, and so on.

This modification of the Newton method has been developed by us in solving a number of problems of rocket dynamics since 1961. We considered two-point boundary-value problems of the 10^{th} to 14^{th} order with as many as six unknown parameters (Bibl.14, 15). Satisfactory results were obtained in a check on this method performed at the All-Union USSR Academy of Sciences by V.N.Lebedev and his associates [cf., for instance, (Bibl.16)].

Other papers (Bibl.15, 17) are devoted to the computational aspects of

Newton's method. One author (Bibl.15) mentions the effect of the accuracy of solution of the Cauchy problem on the convergence of the iteration process, and another author (Bibl.18) describes application of the integrals of the equations of optimum motion to an accuracy check, effect of the modification of Newton's algorithms on convergence far from the neighborhood of the solution, considerations on the limits of applicability of the Kantorovich modification to this problem, etc.. Stensil (Bibl.17) in particular, discusses a highly interesting method of correcting the Jacobi matrices for the values of the function ϕ.

3. Brief Survey of the Results Obtained by the Newton Method /13

Newton's method has proved effective for the numerical solution of a wide range of problems in rocket dynamics: in calculating the program for the angle of thrust orientation of a single-stage rocket with maximum velocity at apogee (Bibl.14); in the search for optimum programs of variation of power, exhaust velocity, and thrust orientation of a space vehicle on an interorbital flight to Mars with limited power $0 \le N \le N_{max}$ and limited variational range of exhaust velocity $C_{min} \le C \le C_{max}$ (Bibl.14, 15). Of the results obtained in one of these papers (Bibl.14) we particularly note (Fig.1) the evaluation of the influence of restrictions of the exhaust velocity on the power consumption during an interorbital flight to Mars lasting 0.5 year (the heliocentric angle of flight being π). That paper also presents an example of asymmetric flight with return (Earth-Mars-Earth, total angle of flight 2π, duration 1 year), giving a 20% advantage, in the sense of $\int_0^T a^2 dt$ over symmetrical flight (on which the heliocentric angle and time of flight are the same, going and returning).

We present some results of a numerical study of the effect of the parameter $\tilde{N} = \frac{N_{max}}{2M(0)}$, [where $M(0)$ is the initial mass] on the final mass at the prescribed values $C_{min} = 20$ km/sec, $C_{max} = 100$ km/sec, with respect to the characteristics of an optimal Earth-to-Mars flight ($T = 0.5$ year). It is /14 intuitively clear that there exists a minimum value $\tilde{N}_{min}$ at which the flight will be accomplished at maximum thrust [$c(t) = C_{min}$, $0 \le t \le T$]. For $\tilde{N} < N_{min}$ there is no solution satisfying the boundary conditions (the boundary is arbitrarily shown by the broken line in Fig.2). As $\tilde{N}$ increases, all the types of optimum regimes described in another paper (Bibl.14) occur on the trajectories. Finally, beginning at some value of $\tilde{N}^*$, the motion is accomplished without regulation of the exhaust velocity ($C = C_{max}$) and the length of the passive phase increases (Fig.2). Figure 3 shows the dependence of the final mass on the parameter $\tilde{N}$, under the same restrictions. It is well known that for an ideally controllable system ($C_{min} = 0$, $C_{max} = \infty$), the problem of optimization is resolved into two independent problems: 1) to find the optimum trajectory from the minimum condition $S = \int_0^T a^2 dt$ (the trajectory part of the problem); 2) to find the program of mass consumption $m(t) = m_0 : \left(1 + \frac{m_0}{\tilde{N}} \int_0^t a^2 d\xi\right)$ and optimization of the weight ratios: final mass to parameter $\tilde{N}$, from the value found for S (the weight part of the problem). Figure 3 gives the values of the final mass m(T) for an ideal system [determined by the formulas given in another paper (Bibl.18, 19)] and those found by the numerical method for a flight with the above characteristics. The existence of constraints (C_{min}, C_{max}) leading to the appearance of passive phases will, in the general case, obviously make it impossible to separate the problem of optimization into two independent parts. Thus, in the general case, instead of calculating m(T) and $\tilde{N}$ from the final /15 relations as a function of $\int_0^T a^2 dt$, use should be made of relations of the type shown in Fig.3.

Newton's method has been used in solving problems of the launching of a space vehicle from a circular orbit (Bibl.20), of the flight between two points in a central field (Bibl.21), and of a flight between heliocentric orbits by the aid of a solar sail (Bibl.22).

The application of Newton's method in the work of American specialists is described in Sect.21 of the review by Grodzovskiy et al (Bibl.23).

4. On the Connection between Newton's Method and the Method of Steepest Descent

Practical work has shown that the "cycling" of the iteration process sometimes involves a tendency of α^* to vanish, which is equivalent to an interruption of the descent (and to a decrease of the residue).

Let us rewrite eqs.(7) and (8) in the following form:

$$\Phi = \varphi_* \|A\| \varphi \tag{7a}$$

and

$$\psi = \psi(\varphi_* \|A\| \varphi), \tag{8a}$$

where φ_* is a row vector and $\|A\|$ the symmetric matrix of a positive-determinate quadratic form; φ is a column vector.

Let us give the vector y the increment Δy. Then, the function Φ will take the increment

$$\Delta \Phi = 2\varphi_* \|A\| \|F\| \Delta y + O(|\Delta y|^2), \tag{11}$$

where Δy is a column vector of the increment, and

$$\|F\| = \left\| \begin{matrix} \frac{\partial \varphi_1}{\partial y_1}, & \dots, & \frac{\partial \varphi_1}{\partial y_m} \\ \dots & \dots & \dots \\ \frac{\partial \varphi_m}{\partial y_1}, & \dots, & \frac{\partial \varphi_m}{\partial y_m} \end{matrix} \right\| \tag{12}$$

If there exists a vector η_k satisfying the equation

$$\|F^{(k)}\| \eta_k = -\varphi_k , \tag{13}$$

then, obviously, the equation

$$\|F^{(k)}\| \alpha \eta_k = -\alpha \varphi_k \tag{14}$$

will be satisfied for any α.

Substituting eq.(14) into eq.(11), we obtain

$$\Delta \Phi = -2\varphi_* \|A\| \varphi \alpha + O(|\alpha\eta|^2). \tag{15}$$

Similarly,

$$\Delta \Psi = -\Psi'(\varphi_* \|A\| \varphi) \cdot 2\varphi_* \|A\| \varphi \alpha + O(|\alpha\eta|^2). \tag{16}$$

It follows from eqs.(15) and (16) that, for sufficiently small α, the increments $\Delta\Phi$ and $\Delta\Psi$ will be defined:

$$d\Phi = 2\varphi_* \|A\| \|F\| dy = -2\alpha\varphi_* \|A\| \varphi = -2\alpha\Phi \tag{17}$$

and /17

$$d\Psi = \Psi'(\varphi_* \|A\| \varphi) d\Phi = -2\alpha \Psi'(\varphi_* \|A\| \varphi) \Phi . \tag{18}$$

By virtue of the positive determinacy of Φ, of the conditions imposed on Ψ, and of the conditions of existence of a solution of eqs.(13) or (14),

$$d\Phi < 0 ; \quad d\Psi < 0 . \tag{19}$$

This proves that, at each step of the iteration, there exists an α_k^*, $0 < < \alpha \leq 1$, such that the sequence of levels of the functions of residue arranged in order of the number of the iteration step decreases monotonically:

$$\Phi(y^{k+1}) < \Phi(y^k),$$

where

$$y^{k+1} = y^k + \alpha_k^* \eta^k.$$

The geometrical interpretation of conditions (17), (18), and (19), or

$$(-\text{grad}\,\Phi)\cdot\eta = 2\Phi, \qquad\qquad (20)$$
$$(-\text{grad}\,\Psi)\cdot\eta = 2\Psi'\Phi$$

is that the vector η, at every point [except that of the solution (6)] has a positive direction on the direction of decline of the residue function, i.e., in other words, that η is nowhere tangent to the surface of the level ϕ or ψ but makes an angle less than $\frac{\pi}{2}$ with the direction of most rapid decline ϕ.

The latter considerations have all been based on the assumption that the relevant calculations have been absolutely accurate. In reality, it is precisely this that cannot be assured for numerical calculations on a digital computer. /18

Owing to errors in computation, the angle between the directions of $-\text{grad}\,\phi$ and the vector η may reach or exceed $\frac{\pi}{2}$. If this takes place, then the sequence $\phi(y^k)$ ceases to be monotonic in k, so that at some k* there is a "contraction" of the working point to its initial position:

$$\alpha_{k^*}^* = 0, \quad y^{k^*+1} = y^{k^*}$$

In this case, in using the algorithms of the modified Newton's method (Bibl.8, 14), we can recommend: a) increasing the accuracy of computation (the accuracy of the solution of the Cauchy problem for determining the vector φ, and the accuracy of computation of the Jacobi matrix); b) changing to the method of steepest descent, i.e., to determine the increment dy not by the formula $dy = \alpha\eta$ but rather by the formula $dy = -\alpha\,\text{grad}\,\phi$, or $dy = -\alpha\,\text{grad}\,\psi$, searching for the value of α^* corresponding to this choice.

Up to now, we started out from the existence of a solution η^k for eqs.(13). Since the iteration is performed from the point y^k, which is not a root of eq.(6), the case when eq.(13) is an inhomogeneous system is of interest here. The condition of unsolvability of such a system is that the Jacobi matrix $\|F\|$ shall vanish:

$$|F| = 0 \tag{21}$$

In the general case, eq.(21) is the equation of an (m-1)-dimensional surface in an m-dimensional space($y_1, \ldots, y_m$). The following assertion is valid: The vector of grad ϕ (or of grad ψ) vanishes at the point y, not being a solution of eq.(6), if and only if y belongs to the surface (21). /19

From eqs.(17) and (18) we have

$$\text{grad}\, \Phi = 2\varphi_* \|A\| \|F\| \tag{22}$$

and

$$\text{grad}\, \Psi = \Psi'(\varphi_* \|A\| \varphi) \cdot 2\varphi_* \|A\| \|F\| \cdot \tag{23}$$

By virtue of the condition $\psi^1 > 0$, grad ϕ and grad ψ vanish simultaneously, so that we will confine ourselves to a consideration of the vector of grad ϕ. By hypothesis, φ_* does not vanish, and $\|A\|$ is the matrix of the positive-determinate quadratic form for which the Sylvester criterion is valid, whence, in particular, it follows that its determinant does not vanish, i.e., that the system of linear equations

$$\varphi_* \|A\| = 0_*,$$

where 0_* is a zero row vector, has only the trivial solution $\varphi_* = 0_*$.

By virtue of the contradiction with the condition we conclude that the vector

$$\mu = \varphi_* \|A\| \neq 0_*.$$

We have, further:

$$\mu \|F\| = 0_*. \tag{24}$$

Since μ is a nonzero vector, eq. (24) can mean only that $|F| = 0$, which was to be proved.

The assertion just obtained is a necessary (but not a sufficient) condition of the vanishing of grad ϕ. Thus, it is possible to adduce the example of a surface (21) on which grad ϕ vanishes at a single point. /20

This result permits posing a new problem: that of the existence (and search for) a certain function ψ, optimum for an assigned space Y [i.e., in the language of Gel'fand (Bibl.9), that of improving the organization of the function ψ]. In any case, on the basis of the result obtained, we can hope to introduce several functions $\psi_1, \ldots, \psi_s$, such that the points on these, where grad $\phi_1 = 0, \ldots,$ grad $\phi_s = 0$, are assumed not to coincide except for the point at which $\phi_1 = \ldots = \phi_s = 0$. This problem and all its aspects remain unsolved (for example, qualitative correspondence, in the sense of change of level of each of the residues when the working point moves in the parameter space).

A comparative analysis of Newton's method and the method of steepest descent, together with considerations on their use in the problem of determining the parameters of the trajectory from measurement data, will be found elsewhere (Bibl.24).

5. The Method of Steepest Descent

While Newton's method is based on the solution of the system (13), the method of steepest descent leading to the decrease of the function $\phi = \sum_{i=1}^{n} \varphi_i^2$ reduces to one of the methods of integrating the system of ordinary differential equations (Bibl.10, 24 - 27)

$$\frac{dy_i}{ds} = -\frac{\frac{\partial \Phi}{\partial y_i}}{\sqrt{\sum_{k=1}^{n}\left(\frac{\partial \Phi}{\partial y_k}\right)^2}} \quad (k = 1, \ldots, n) \tag{25}$$

where the variable S is the parameter along the curve of descent in the m-dimensional space $Y = \{y_1, \ldots, y_m\}$ [so that to each S there corresponds $y_i = y_i(S)$, $i = 1, \ldots, m$] with initial data corresponding to the zero-th approximation of the variables y_i. Effective variants of the method of steepest descent, developed in 1957, are discussed in another paper (Bibl.25, 26) in their application to the problem of working up the astronomical observations of the first artificial satellites, later generalized to the case of problem-solving in determining the orbits of spacecraft flying to the moon or the planets of the solar system. A similar method was proposed by Gavurin (Bibl.27). /21

A variant of the method of steepest descent (Bibl.25 - 27), likewise designed to overcome the computational difficulties connected with the complex relief of the function $\phi(y_1, \ldots, y_m)$ and with the strong inhomogeneity of the slope of the descent for the individual variables, is based on the following transformation of the variables y_i to new variables φ_i:

$$\varphi_i = \varphi_i(y_1, \ldots, y_m), \; (i = 1, \ldots, m). \tag{26}$$

The equations of motion on the gradient lines (25) in the new variables are greatly simplified, and take the form:

$$\frac{d\varphi_i}{ds} = -\frac{\varphi_i}{\sqrt{\Phi}}, \quad (i = 1, \ldots, m). \tag{27}$$

The solution of eq.(27) is obtained in the form of

$$\varphi_i(s) = \varphi_i^\circ\left(1 - \frac{s}{\sqrt{\Phi^\circ}}\right), \quad (i = 1, \ldots, m). \tag{28}$$

where /22

$$\varphi_i^\circ = \varphi_i(y^\circ); \quad \Phi^\circ = \sum_{k=1}^{m} [\varphi_k^\circ]^2,$$

$y^\circ = (y_1^\circ, \ldots, y_m^\circ)$ is the zero-th approximation

Thus, motion in the subspace of the new variables (as which we select the components of the function of total residue ϕ) proceeds along the straight lines (28), i.e., in a certain sense most "rapidly" toward the solution sought $\phi = 0$. The actual search for the initial variables is performed by the aid of the system (27), which may be transformed into the following form:

$$a_{i1}\frac{dy_1}{ds} + a_{i2}\frac{dy_2}{ds} + \cdots + a_{im}\frac{dy_m}{ds} = -\frac{\varphi_i^\circ}{\sqrt{\Phi^\circ}}, \tag{29}$$

where

$$a_{ij} = \frac{\partial \varphi_i}{\partial y_j}, \quad (i, j = 1, \ldots, m). \tag{30}$$

Consequently, this variant of the steepest descent reduces to the integration of a system of the form of eqs.(27) over the interval $0 \le s \le \sqrt{\phi^0}$. The method is readily generalized to the case when the number of equations $\varphi_i = 0$ exceeds the number of unknown parameters y_i, which occurs in the statistical work-up of an excess number of measurements.

6. Methods of Solving Special Problems /23

The special problems include the extensive practical class of problems with a free right end of the trajectory (Bibl.28, 29). Without limitation of generality, let us consider the problem with the fixed time T (in a problem with free time, there is one condition that determines the end of the process, and all the procedure to be described below may be applied without trouble to this case).

Let us describe the algorithm (Bibl.28):

1) Assign as first approximation (for instance, from physical considerations) a certain allowable control $u^{(1)}(t)$, $0 \le t \le T$. 2) Substituting the control $u^{(1)}(t)$ for u in the system (1a), we integrate this system for the initial conditions (2) over the interval $[0, T]$, and denote the solution so obtained by $x^{(1)}(t)$. 3) Substituting $u^{(1)}(t)$ and $x^{(1)}(t)$ into the second part of the system (4):

$$\frac{dp_i}{dt} = -\frac{\partial H(x, u, p, t)}{\partial x_i}, \quad (i = 1, \ldots, n), \tag{31}$$

we integrate eqs.(31) "from right to left" (from T to 0) under the initial conditions (2)

$$p_i(T) = -C_i, \tag{32}$$

where C_i are coefficients in the functional $S = \sum_{i=1}^{n} C_i x_i(T)$ of the Mayer problem. Let us denote the resultant solution by $p^{(1)}(t)$. /24

4) Substituting $x^{(1)}(t)$, $p^{(1)}(t)$ into eq.(6), we then express H as a function of u and t. We now determine the next approximation for the control $u^{(2)}(t)$ from the maximum principle:

$$H(x^{(1)}(t), p^{(1)}(t), u^{(2)}(t), t) = \max_{u \in U} H(x^{(1)}(t), p^{(1)}(t), u, t)$$

5) Using $u^{(2)}(t)$, we find successively $x^{(2)}(t)$, $p^{(2)}(t)$, etc. If the process of successive approximations is convergent, we continue it until the successive approximations differ by no more than the allowable accuracy limit.

The above-described algorithm is computationally simple, i.e., at each step it reduces to the solution of two Cauchy problems: "from left to right" for the system (1) and "from right to left" for the system (31).

For the case of linear systems in which $f = A(t)x + b(t)u$, application of this algorithm gives an exact solution in the second approximation (the exact values of the conjugate variables and the optimum control being determined in the first approximation, and the coordinates in the second).

Krylov (Bibl.28) also discusses using the method on a digital computer, with special reference to the economy of the machine memory. As an illustration of the application of the method, the authors consider the model problem of programming the angle of attack and the choice of one of two discrete values of the characteristic value during the flight of a material point to maximum range through a resisting medium.

The problem of the flight of a vehicle with an engine to maximum range /25
in an atmosphere has been reduced (Bibl.29) to the problem with a free right

end of the trajectory. Under certain assumptions, the value of the vector p(T) is completely determined, so that the values of the phase coordinates at the time t = T are the unknown parameters. The above author organizes the search for these on the basis of Newton's method, integrating the system (4) "from right to left".

7. Linear Systems

The optimum processes of flight dynamics are described by essentially nonlinear differential equations. Nevertheless, we should like here to call the reader's attention to one interesting numerical method of solving problems of optimal control - a method which in its original form was developed for linear systems (Bibl.30, 31) and was subsequently generalized to other classes of problems (Bibl.32).

This method reduces the problem of finding the optimal control u(t) that will minimize the coordinate $x_0(T)$ under the condition $\tilde{x}(T) = x_1^1, \ldots, x_n^1) = \tilde{x}_1$ to the problem of finding a reference hyperplane to the set M(T) of points of an (n+1)-dimensional space to which the system (1) can be transferred by the aid of the allowable control u(t), $0 \leq t \leq T$. Under the assumption (Bibl.1) of "strict convexity of M(T)", the assigned hyperplane to the set M(T) (the existence of which follows from the maximum principle) uniquely determines the control and thus also the point $X(T) = \{x_0(T), \tilde{x}_1(T)\}$ of the (n+1)-dimensional /26
space. To find the reference hyperplane to the set M(T) at the point of intersection of the set M(T) and the straight line $x = \tilde{x}_1$, with the smallest coordinate x_0, a process of successive approximation has been proposed (Bibl.30-32). A feature of this approach is that the process of successive approximation to the initial hypersurface is monotonic (Bibl.32).

This approach may also be of interest in connection with the computational procedure proposed by Bellman and Kalaba (Bibl.33), which reduces the solution of the boundary-value problem for a nonlinear system of differential equations to the solution of the analogous problem for a sequence of linear equations which, under certain assumptions, converge to the solution of the nonlinear system.

II. DIRECT METHODS /27

The direct methods have been extensively discussed in numerous papers (Bibl.10, 23, 34), which give exhaustive bibliographical references to which the reader is referred.

Here we merely note the main trends in the development of methods of this type:

1. The method of the functional of steepest descent (Bibl.10), employed in the problem of minimization of $\int_0^T a^2 dt$ for a flight between heliocentric orbits

(Bibl.35).

2. The gradient method (Bibl.34, 36 - 39).

3. The Ritz method (Bibl.24, 35).

[In solving the problem of the minimum of $\int_0^T a^2 dt$, Ivanov (Bibl.35) notes that the efficiency of the method of the descent functional is substantially greater (by one or two orders of magnitude) than that of the Ritz method, in which the coefficients are determined by the method of steepest descent].

4. The broken-lines method (Bibl.40).

A review of these methods would be beyond the scope of this paper.

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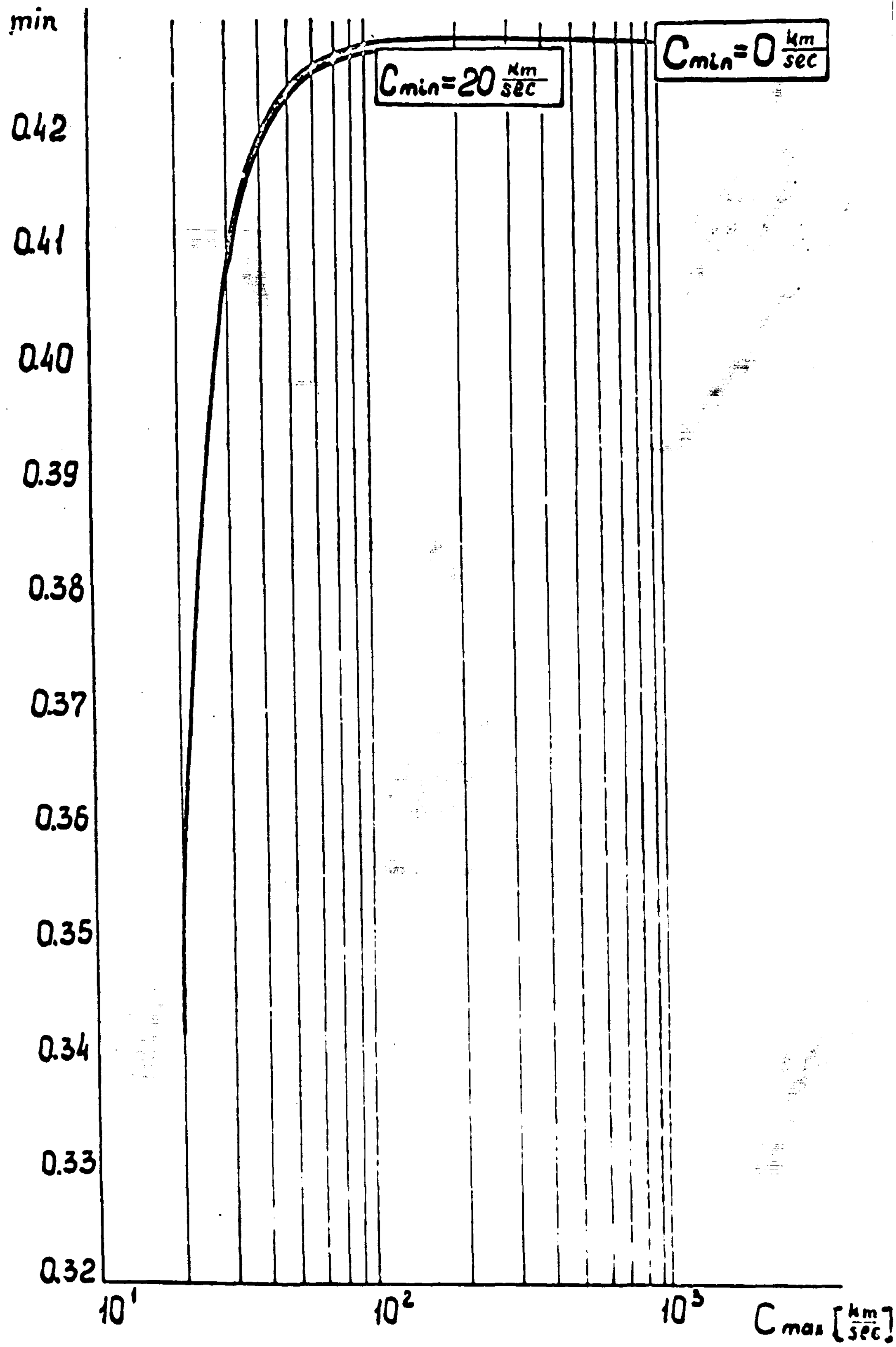
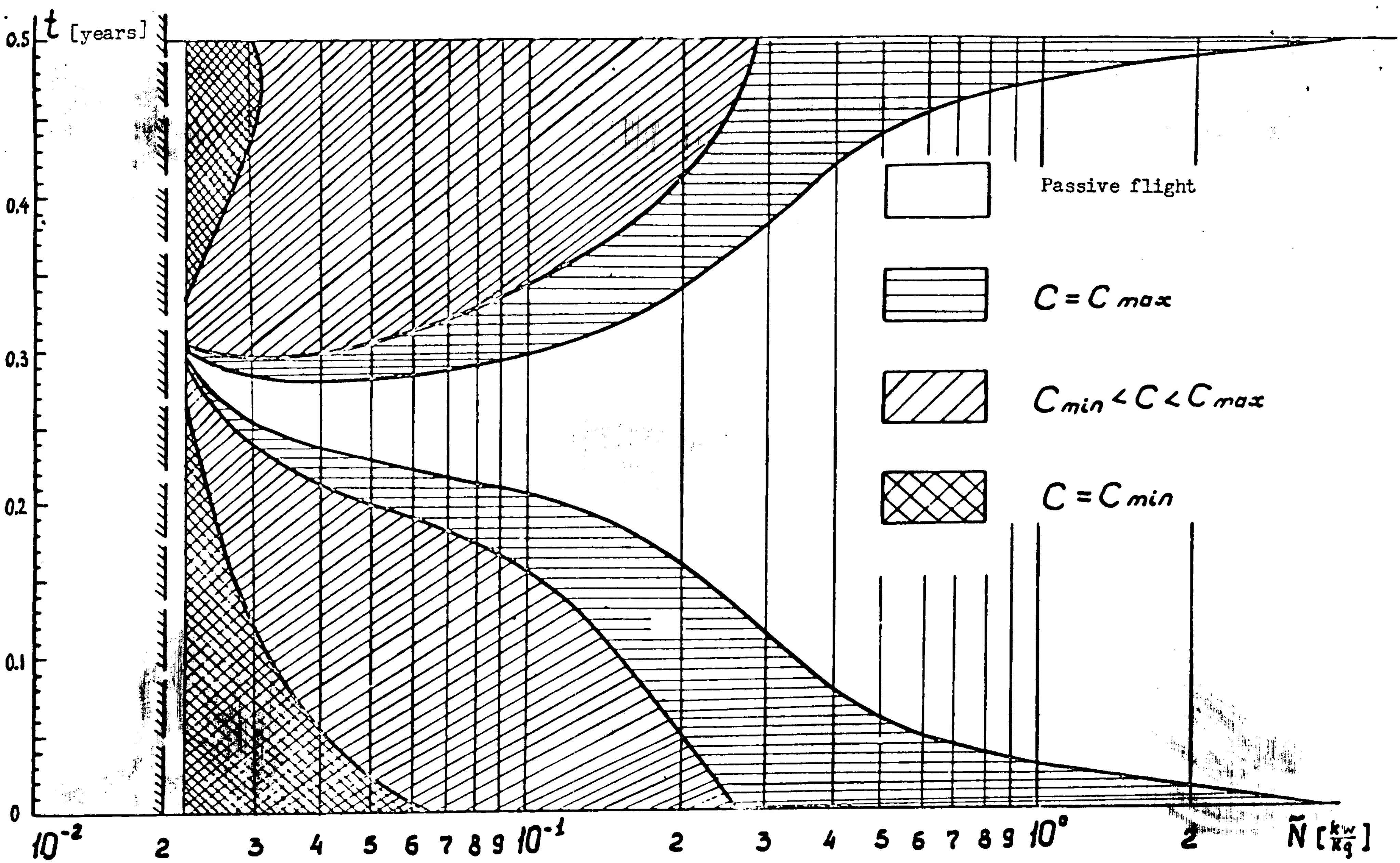


Fig.1 Earth-Mars Flight (T = 0.5 km)



t [years]
0.5
0.4
0.3
0.2
0.1
0
Passive flight
$C = C_{max}$
$C_{min} < C < C_{max}$
$C = C_{min}$
10^{-2} 2 3 4 5 6 7 8 9 10^{-1} 2 3 4 5 6 7 8 9 10^{0} 2
$\tilde{N}\ [\frac{kw}{kg}]$

Fig.2

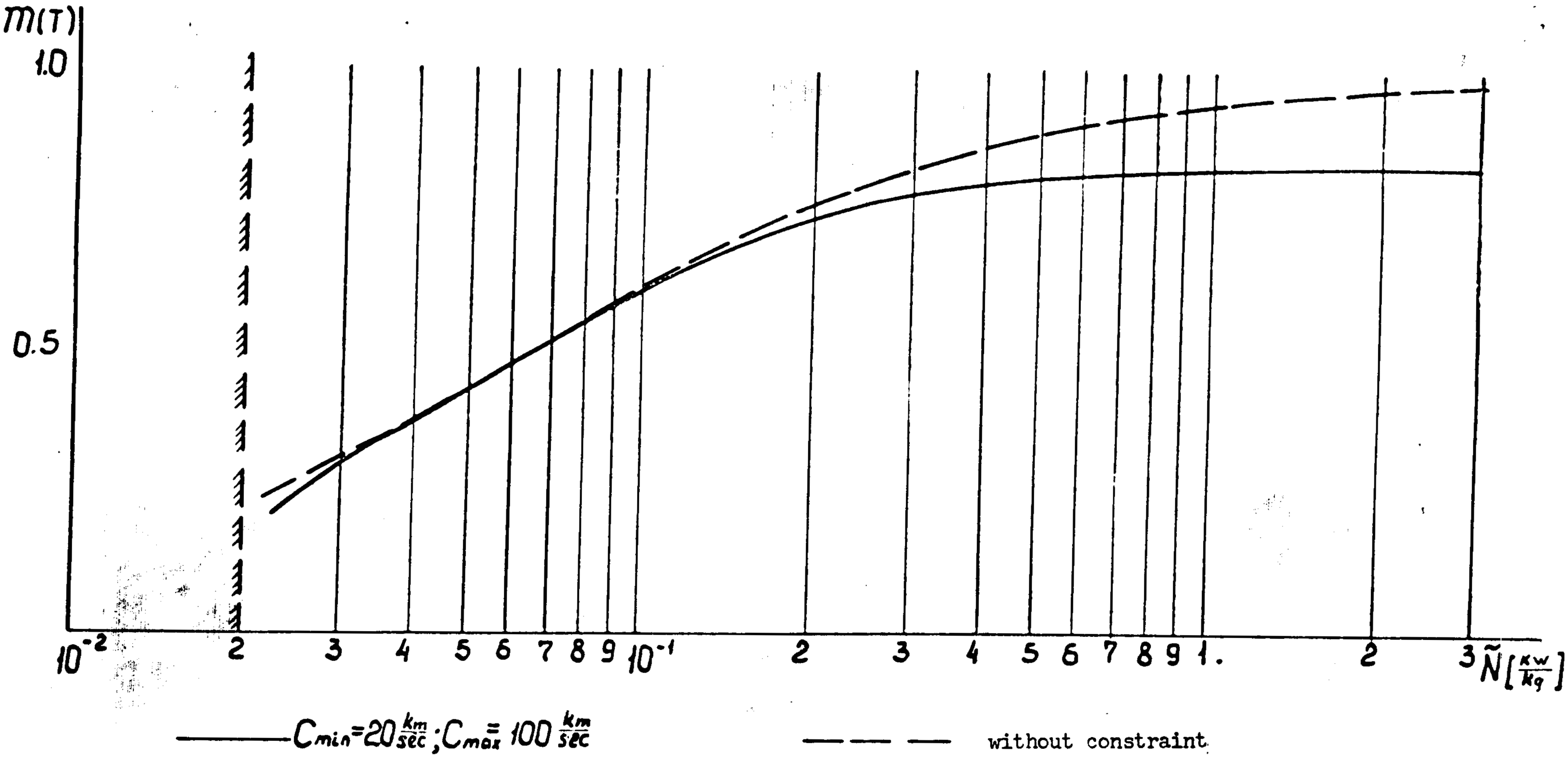


Fig.3

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